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Application of the exact quantization rule to the relativistic solution of the rotational Morse potential with pseudospin symmetry

Wen-Chao Qiang¹, Run-Suo Zhou² and Yang Gao³

¹ Faculty of Science, Xi'an University of Architecture and Technology, Xi'an, 710055, People's Republic of China

² Department of Nuclear Medicine, TangDu Hospital, The Fourth Military Medical University, Xi'an 710038, People's Republic of China

³ Xi'an Microelectronics Technology Institute, Xi'an 710054, People's Republic of China

E-mail: qwcqj@pub.xaonline.com

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Abstract

We present an analytical solution of the radial Dirac equation for the rotational Morse potential through the Pekeris approximation. The bound state energy eigenvalues are obtained by using an exact quantization rule for non-zero κ values of the Dirac equation. As an application of the rule, we give the numerical solutions of the results for special values of the potential parameters.

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1. Introduction

The Morse potential has attracted a great deal of interest as an important and basic model for describing the interaction between two atoms in a diatomic molecule [1–7]. It is well known that there are no analytical solutions for the rotational Morse potential for the Schrödinger equation with $l \neq 0$ or for the Dirac equation with $\kappa \neq 0$, so various approximations are employed to derive numerical or quasi-analytical solutions. For example, Killingbeck *et al* [4] treat the full problem of the Morse potential with angular momentum. Their method first finds the minimum $x = x_0$ of the effective potential V , then forms the Taylor series expansion of V at the minimum. Finally, hypervirial perturbation is applied to obtain the numerical eigenvalues [4]. This method can only be used to get a numerical solution, and for a quasi-analytical solution an efficient approximate method, the Pekeris approximation is widely used. Recently, Morales, Bayrak *et al* and Berkdemir all employed the frame of the Pekeris approximation, but with different approaches. Morales used the supersymmetric quantum mechanics (SQM) approach [8], Bayrak the asymptotic iteration method (AIM) [9] and Berkdemir the Nikiforov–Uvarov (NU) method [10]. Then, Berkdemir further studied the

relativistic rotational Morse potential with pseudospin symmetry by the Pekeris approximation and the NU method [11]. The pseudospin symmetry results from the near equality in the magnitude of an attractive scalar $-S$ and repulsive vector V , both relativistic mean fields, $S \sim V$, in which the nucleons move. This approach involves a division of the single particle total angular momentum into pseudo rather than normal orbital and spin parts. The concept, $j = \tilde{l} \pm \tilde{s}$, is expressed in terms of a pseudo-orbital angular momentum $\tilde{l} = l + 1$ which is coupled to a pseudo-spin $\tilde{s} = 1/2$ [12–14].

On the other hand, recently, Ma and Xu proposed an exact quantization rule [15, 16] by which they obtained the eigenvalues for a one-dimensional finite square well, the harmonic oscillator potential, Morse potential, asymmetric Rosen–Morse potential, first and second Pöschl–Teller potential, as well as the three-dimensional harmonic oscillator and hydrogen atom. Employing this approach, Qiang and Dong also found arbitrary l -state solutions of the rotating Morse potential with the Pekeris approximation [17]. These results show that the exact quantization rule can be efficiently used to obtain the exact bound-state solutions for most solvable potentials. It must be pointed out that the Hankel quantization condition arising from the Riccati–Padé method also has made much use of the logarithmic approach, but the method is only available for numerical work [18].

Now, although there are no exact quantization rules for the Dirac equation, by transforming it into a Schrödinger-like equation we can still use the exact quantization rule to solve relativistic eigenvalue problems. In this paper we will use Ma’s exact quantization rule to derive arbitrary κ -state solutions to the Dirac equation for the rotational Morse potential with pseudo-spin symmetry within the Pekeris approximation. In the following section, the exact quantization rule is introduced. Then, in section 3, we apply this method to obtain the exact energy eigenvalues. Some details about this study are discussed in section 4. Finally, some concluding remarks are given in section 5.

2. Exact quantization rule

We first give a brief review of the exact quantization rule. Ma *et al* have proved that the one-dimensional Schrödinger equation

$$\frac{d^2}{dx^2} \psi(x) = -\frac{2M}{\hbar^2} [E - V(x)] \psi(x) \quad (1)$$

can be written as

$$\frac{d}{dx} \phi(x) = -\frac{2M}{\hbar^2} [E - V(x)] - \phi(x)^2, \quad (2)$$

where $\phi(x) = \psi(x)^{-1} d\psi(x)/dx$ is the logarithmic derivative of the wavefunction $\psi(x)$, M represents the mass of the particle and the potential $V(x)$ is a piecewise continuous real function of the x . Furthermore, it is known from the Sturm–Liouville theorem that $\phi(x)$ decreases monotonically with respect to x between two turning points, where $E \geq V(x)$. Specifically, as x increases across a node of the wavefunction $\psi(x)$, where $E \geq V(x)$, $\phi(x)$ decreases to $-\infty$, jumps to $+\infty$, and then decreases again. By carefully studying the one-dimensional Schrödinger equation they proposed an exact quantization rule

$$\int_{x_A}^{x_B} k(x) dx = N\pi + \int_{x_A}^{x_B} \phi(x) \left[\frac{dk(x)}{dx} \right] \left[\frac{d\phi(x)}{dx} \right]^{-1} dx, \quad (3)$$

where $k(x) = \sqrt{2M[E - V(x)]}/\hbar$, x_A and x_B are two turning points determined by $E = V(x)$ and $x_A < x_B$, N is the number of nodes of $\phi(x)$ in the region $E \geq V(x)$ and is larger by 1 than the number of nodes of the wavefunction $\psi(x)$. In addition, the quantization rule (3)

can be easily generalized to the three-dimensional Schrödinger equation with a spherically symmetric potential. For example, if we take $\psi(r) = r^{-1}R(r)Y_m^l(\theta, \phi)$, we are able to obtain the radial Schrödinger equation as

$$\frac{d^2}{dr^2}R(r) = -\frac{2M}{\hbar^2}[E - U(r)]R(r), \quad U(r) = \frac{l(l+1)\hbar^2}{2Mr^2} + V(r). \quad (4)$$

Since equation (4) is similar to equation (1), the quantization rule (3) can be generalized to the three-dimensional Schrödinger equation case

$$\int_{r_A}^{r_B} k(r) dr = N\pi + \int_{r_A}^{r_B} \phi(r) \left[\frac{dk(r)}{dr} \right] \left[\frac{d\phi(r)}{dr} \right]^{-1} dr. \quad (5)$$

In the above equation, the first term $N\pi$ is the contribution from the nodes of the wavefunction, and the second one is called the quantum correction. Ma and Xu have found that this quantum correction is *independent* of the number of nodes of the wavefunction for the exactly solvable systems. Thus, it can be replaced in equation (5) by

$$\int_{r_{0A}}^{r_{0B}} \phi_0(r) \left[\frac{dk_0(r)}{dr} \right] \left[\frac{d\phi_0(r)}{dr} \right]^{-1} dr, \quad (6)$$

where subscript 0 denotes the ground state. Up to now, Ma's exact quantization rule has not yet been extended to the Dirac equation, but we can still use it to solve relativistic eigenvalue problems.

3. Calculation of the energy eigenvalues by the exact quantization rules and the Pekeris approximation

For simplicity, how to derive the radial equation of the Dirac equation with pseudospin symmetry is ignored since a detailed discussion can be found in [11–13]. The radial parts of the Dirac equation with spherical symmetry are a set of two coupled ordinary differential equations

$$\begin{aligned} \left(\frac{d}{dr} + \frac{k}{r} \right) F_{nk}(r) &= [\mu + E_{nk} - \Delta(r)] G_{nk}(r), \\ \left(\frac{d}{dr} - \frac{k}{r} \right) G_{nk}(r) &= [\mu - E_{nk} + \Sigma(r)] F_{nk}(r), \end{aligned} \quad (7)$$

where $k = \pm(j+1/2)$ is the eigenvalue of $\mathbf{K} = -\beta(\vec{\sigma} \cdot \vec{L} + 1)$, j is the total angular momentum of the system under consideration, E_{nk} is the energy eigenvalue, $\Delta(r) = V(r) - S(r)$, $\Sigma(r) = V(r) + S(r)$, and $V(r)$ and $S(r)$ are the repulsive vector potential and attractive scalar potential, respectively. Equation (7) can be decoupled to two second-order differential equations for F_{nk} and G_{nk} , respectively,

$$\begin{aligned} \left(\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - (\mu + E_{nk} - \Delta(r)) \left(\mu - E_{nk} + \Sigma(r) \right) - \frac{\frac{d\Sigma}{dr} \left(\frac{d}{dr} - \frac{k}{r} \right)}{\mu - E_{nk} + \Sigma(r)} \right) G_{nk}(r) &= 0, \\ \left(\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (\mu + E_{nk} - \Delta(r)) \left(\mu - E_{nk} + \Sigma(r) \right) + \frac{\frac{d\Delta}{dr} \left(\frac{d}{dr} + \frac{k}{r} \right)}{\mu + E_{nk} - \Delta(r)} \right) F_{nk}(r) &= 0. \end{aligned} \quad (8)$$

In the case of exact pseudospin symmetry when $d\Sigma/dr = 0$, i.e., $\Sigma = C = \text{const}$, the equation about $G_{nk}(r)$ becomes

$$\left(\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} + (\mu - E_{nk} + C)\Delta(r) + (E_{nk}^2 - \mu^2 - C(\mu + E_{nk})) \right) G_{nk}(r) = 0. \quad (9)$$

In this work we assume that $\Delta(r)$ takes the form of the Morse potential,

$$U(r) = D(e^{-2a(r-r_0)} - 2e^{-a(r-r_0)}), \quad (D > 0, a > 0), \quad (10)$$

where D is the dissociation energy, r_0 is the equilibrium distance (bond length) and a is a parameter to control the width of the potential well. Generally speaking, equation (9) with $\Delta(r) = U(r)$ has no analytical solution for $k \neq 0$. Therefore, an approximation has to be made. The most widely used and convenient one is the Pekeris approximation. This approximation is based on the expansion of the centrifugal barrier in a series of exponentials depending on the internuclear distance, keeping terms up to second order. It should be pointed out, however, that this approximation is valid only for low vibrational energy states. In the Pekeris approximation, the centrifugal potential can be approximately written as [11]

$$\tilde{V}_k(r) = \frac{k(k-1)}{r^2} = \gamma(D_0 + D_1 e^{-\alpha x} + D_2 e^{-2\alpha x}), \quad (11)$$

with

$$D_0 = 1 - \frac{3}{\alpha} + \frac{3}{\alpha^2}, \quad D_1 = \frac{4}{\alpha} - \frac{6}{\alpha^2}, \quad D_2 = -\frac{1}{\alpha} + \frac{3}{\alpha^2}, \quad (12)$$

and

$$\gamma = \frac{k(k-1)}{r_0^2}, \quad \alpha = ar_0, \quad x = \frac{r-r_0}{r_0}. \quad (13)$$

From equations (11)–(13), equation (9) now becomes

$$\left(\frac{d^2}{dx^2} - r_0^2 \gamma (D_0 + D_1 e^{-\alpha x} + D_2 e^{-2\alpha x}) + v^2 (e^{-2\alpha x} - 2e^{-\alpha x}) + \omega^2 \right) G_{nk}(x) = 0, \quad (14)$$

where

$$v^2 = r_0^2 D(\mu - E_{nk} + C), \quad \omega^2 = r_0^2 (E_{nk}^2 - \mu^2 - C(\mu + E_{nk})). \quad (15)$$

After introducing the following notation

$$\begin{aligned} \epsilon &= -\sqrt{r_0^2 D_0 \gamma - \omega^2}, \\ \beta_1 &= \sqrt{-(r_0^2 D_1 \gamma + 2v^2)}, \\ \beta_2 &= \sqrt{r_0^2 D_2 \gamma - v^2}, \\ V_{\text{eff}}(x) &= -(\beta_1^2 e^{-\alpha x} - \beta_2^2 e^{-2\alpha x}), \end{aligned} \quad (16)$$

equation (14) can be expressed as

$$\left(\frac{d^2}{dx^2} - (\epsilon^2 + \tilde{V}_{\text{eff}}(x)) \right) G_{nk}(x) = 0. \quad (17)$$

Now we can apply the quantization rule in equation (5) to the effective potential $\tilde{V}_{\text{eff}}(x)$. For this purpose, we first solve the Riccati equation

$$\frac{d}{dx}\phi_0(x) = -(-\epsilon_0^2 - \tilde{V}_{\text{eff}}(x)) - \phi_0(x)^2. \tag{18}$$

Introducing a new variable of the form $y = e^{-\alpha x}$ and using equation (16), we obtain

$$\alpha y \frac{d}{dy}\phi_0(y) - (-\epsilon_0^2 - \tilde{V}_{\text{eff}}(y)) - \phi_0(y)^2 = 0. \tag{19}$$

We note that $r \in (0, \infty)$, $x \in (-1, \infty)$ and $y \in (e^\alpha, 0)$; therefore while $\phi_0(x)$ decreases as x increases, $\phi_0(y)$ increases as y increases. The solution with one node and no pole only has the form $\phi_0(y) = ay + b$, where $a > 0$ due to the monotonic property. Substituting this trial solution into the Riccati equation (19) we obtain

$$\epsilon_0 = \pm \frac{1}{2} \left(\alpha - \frac{\beta_1^2}{\beta_2} \right), \quad b = -\frac{\beta_1^2 - \alpha\beta_2}{2\beta_2}, \quad a = \beta_2 > 0. \tag{20}$$

Hence

$$\begin{aligned} \phi_0(y) &= \beta_2 y - \frac{\beta_1^2 - \alpha\beta_2}{2\beta_2}, \\ \tilde{V}_{\text{eff}}(y) &= -(\beta_1^2 y - \beta_2^2 y^2), \\ k_0(y) &= \sqrt{-\epsilon_0^2 - \tilde{V}_{\text{eff}}(y)} = \beta_2 \sqrt{(y_{0a} - y)(y - y_{0b})}, \end{aligned} \tag{21}$$

where

$$y_{0a} = \frac{\beta_1^2 + \sqrt{\alpha\beta_2(2\beta_1^2 - \alpha\beta_2)}}{2\beta_2^2}, \quad y_{0b} = \frac{\beta_1^2 - \sqrt{\alpha\beta_2(2\beta_1^2 - \alpha\beta_2)}}{2\beta_2^2}. \tag{22}$$

We can now calculate the quantum correction,

$$\begin{aligned} &\int_{x_{0a}}^{x_{0b}} \phi_0(x) \left[\frac{dk_0(x)}{dx} \right] \left[\frac{d\phi_0(x)}{dx} \right]^{-1} dx \\ &= \frac{1}{\alpha} \int_{y_{0b}}^{y_{0a}} \frac{1}{y} \phi_0(y) \left[\frac{dk_0(y)}{dy} \right] \left[\frac{d\phi_0(y)}{dy} \right]^{-1} dy \\ &= \frac{1}{4\alpha} \int_{y_{0b}}^{y_{0a}} \frac{-4\beta_2 y^2 + 2y \left[-\alpha + \frac{\beta_1^2}{\beta_2} + (y_{0a} + y_{0b})\beta_2 \right] + \frac{(y_{0a} + y_{0b})(-\beta_1^2 + \alpha\beta_2)}{\beta_2}}{y\sqrt{(y_{0a} - y)(y - y_{0b})}} dy \\ &= -\frac{\beta_2}{\alpha} \int_{y_{0b}}^{y_{0a}} \frac{y}{\sqrt{(y_{0a} - y)(y - y_{0b})}} dy \\ &\quad + \frac{-\alpha + \frac{\beta_1^2}{\beta_2} + (y_{0a} + y_{0b})\beta_2}{2\alpha} \int_{y_{0b}}^{y_{0a}} \frac{1}{\sqrt{(y_{0a} - y)(y - y_{0b})}} dy \\ &\quad + \frac{(y_{0a} + y_{0b})(-\beta_1^2 + \alpha\beta_2)}{4\alpha\beta_2} \int_{y_{0b}}^{y_{0a}} \frac{1}{y\sqrt{(y_{0a} - y)(y - y_{0b})}} dy \\ &= \frac{\pi(-y_{0a} - y_{0b} + 2\sqrt{y_{0a}y_{0b}})(\beta_1^2 - \alpha\beta_2)}{4\alpha\beta_2\sqrt{y_{0a}y_{0b}}} \\ &= \pi \left(-\frac{1}{2} + \frac{\beta_1^2}{\alpha\beta_2} \right), \end{aligned} \tag{23}$$

where the condition $\beta_2 > 0$, $\beta_1^2 - \alpha\beta_2 < 0$ and following formulae have been used:

$$\begin{aligned}\int_a^b \frac{y}{\sqrt{(y-a)(b-y)}} dy &= \frac{\pi}{2}(a+b), \\ \int_a^b \frac{1}{\sqrt{(y-a)(b-y)}} dy &= \pi, \\ \int_a^b \frac{1}{y\sqrt{(y-a)(b-y)}} dy &= \frac{\pi}{\sqrt{ab}}.\end{aligned}\quad (24)$$

Second, we calculate the left side of equation (5) for the effective potential $\tilde{V}_{\text{eff}}(y)$. For simplicity, we write $k(y)$ as follows:

$$K(y) = \sqrt{-\epsilon^2 + \beta_1^2 y - \beta_2^2 y^2} = \beta_2 \sqrt{(y_a - y)(y - y_b)}. \quad (25)$$

Here equations (12) and (15) have been used, and $y_a = e^{-\alpha x_a}$ and $y_b = e^{-\alpha x_b}$ are two turning points, determined by $E - \tilde{V}_{\text{eff}}(y) = 0$ and $y_a > y_b$, so that

$$y_{\pm} = \frac{\beta_1^2 \pm \sqrt{\beta_1^4 - 4\beta_2^2 \epsilon^2}}{2\beta_2^2}, \quad (26)$$

where y_{\pm} correspond to y_a and y_b , respectively. Now, it is easy to see that

$$\begin{aligned}\int_{x_a}^{x_b} k(x) dx &= \frac{\beta_2}{\alpha} \int_{y_a}^{y_b} \frac{1}{y} \sqrt{(y_a - y)(y - y_b)} dy \\ &= \frac{\beta_2}{\alpha} \left(\frac{\pi}{2}(y_a + y_b) - \pi \sqrt{y_a y_b} \right) = \frac{\pi(\beta_1^2 + 2\beta_2 \epsilon)}{2\alpha\beta_2},\end{aligned}\quad (27)$$

where the condition $\beta_2 > 0$, $\epsilon < 0$ and following formula have been used:

$$\int_a^b \frac{1}{y} \sqrt{(y-a)(b-y)} dy = \frac{\pi}{2}(a+b) - \pi\sqrt{ab}. \quad (28)$$

Substituting equations (23) and (27) into equation (5), we finally obtain

$$-\frac{\beta_1^2 - 2\beta_2 \epsilon}{\alpha\beta_2} = 2n + 1, \quad (29)$$

where $N = n + 1$ has been used and n is the number of nodes of the wavefunction $G_{nk}(y)$. Using the notation of [11] we obtain

$$\epsilon_1 = \frac{\omega^2}{\alpha^2} - \frac{r_0^2 D_0 \gamma}{\alpha^2}, \quad \epsilon_2 = \frac{2v^2}{\alpha^2} + \frac{r_0^2 D_1 \gamma}{\alpha^2}, \quad \epsilon_3 = \frac{v^2}{\alpha^2} - \frac{r_0^2 D_2 \gamma}{\alpha^2}. \quad (30)$$

It is easy to find the relationship between our notation and that of [11]:

$$\epsilon = -\alpha\sqrt{-\epsilon_1}, \quad \beta_1 = \alpha\sqrt{-\epsilon_2}, \quad \beta_2 = \alpha\sqrt{-\epsilon_3}. \quad (31)$$

Using ϵ_1 , ϵ_2 and ϵ_3 , equation (29) becomes

$$-2\sqrt{-\epsilon_1} + \frac{\epsilon_2}{\sqrt{-\epsilon_3}} = 2n + 1. \quad (32)$$

We note that $\sqrt{-\epsilon_1} = \pm i\sqrt{\epsilon_1}$ and $\sqrt{-\epsilon_3} = \pm i\sqrt{\epsilon_3}$, so to avoid getting an equation which has no solution for energy E we must choose $\sqrt{-\epsilon_1} = -i\sqrt{\epsilon_1}$ and $\sqrt{-\epsilon_3} = -i\sqrt{\epsilon_3}$. This allows equation (32) to be written in the following form:

$$-\left(2\sqrt{\epsilon_1} + \frac{\epsilon_2}{\sqrt{\epsilon_3}}\right) = (2n + 1)i, \quad (33)$$

Table 1. Relativistic energy eigenvalues in units of fm⁻¹ of the pseudospin-symmetry Morse potential using the exact quantization rule method for various values of *n* and *k*. For a special case, *C* = -10 fm⁻¹.

\tilde{l}	$n, k < 0$	(l, j)	$E_{n,k < 0}$	$n - 1, k > 0$	$(l + 2, j + 1)$	$E_{n-1,k > 0}$
1	1, -1	(1s _{1/2})	-0.006 4123	0, 2	(0d _{3/2})	-0.006 4123
2	1, -2	(1p _{1/2})	-0.015 5771	0, 3	(0f _{5/2})	-0.015 5771
3	1, -3	(1d _{1/2})	-0.024 3659	0, 4	(0g _{7/2})	-0.024 3659
4	1, -4	(1f _{1/2})	-0.030 5297	0, 5	(0h _{9/2})	-0.030 5297
1	2, -1	(2s _{1/2})	-0.007 0204	1, 2	(1d _{3/2})	-0.007 0204
2	2, -2	(2p _{3/2})	-0.019 0441	1, 3	(1f _{5/2})	-0.019 0441
3	2, -3	(2d _{5/2})	-0.033 7719	1, 4	(1g _{7/2})	-0.033 7719
4	2, -4	(2f _{7/2})	-0.049 2150	1, 5	(1h _{9/2})	-0.049 2150

which is in agreement with equation (36) of [11]. Substituting the values of ϵ_1, ϵ_2 and ϵ_3 into equation (33), the energy eigenvalue equation for E_{nk} can be immediately obtained:

$$\left(\frac{2D\tilde{E}_{nk} + \gamma D_1}{\sqrt{D\tilde{E}_{nk} - \gamma D_2}} + 2\sqrt{(\tilde{E}_{nk} - C - 2\mu)\tilde{E}_{nk} - D_0\gamma} \right)^2 + (2n + 1)^2 a^2 = 0, \tag{34}$$

where $\tilde{E}_{nk} = \mu + C - E_{nk}$. As Berkdemir [11] pointed out, there are only ‘negative’ energy states, and no bound states with positive energy exist. Imposing appropriate values for the parameters D_i ($i = 0, 1, 2$), C, D and a , we can use any of the equations from (32) to (34) to calculate the negative bound state energies of the pseudospin-symmetry Morse potential for a diatomic molecule. As an example, the numerical solution of equations (32)–(34) with parameters $D = 5.0 \text{ fm}^{-1}, r_0 = 2.408 73 \text{ fm}, a = 0.988 879 \text{ fm}, \mu = 10.0 \text{ fm}^{-1}, D_0 = 0.269 28, D_1 = 0.621 78$ and $D_2 = 0.108 93$ is presented in table 1. We must point out that equations (32) and (33) are both correct energy eigenvalue equations, yet equation (34) is not the same as equation (37) of [11]. The data in table 1 are different from that given in table 1 of that paper. Our equation (33), which is their equation (36), does not agree with their equation (37)

4. Discussion

In this section we give some explanation about this study. First, from table 1 we see that the Dirac eigenstate 1S_{1/2} with $n = 1$ and $k = -1$ has a partner which is denoted by 0d_{3/2} with $n - 1 = 0$ and $k = 2$. The two eigenstates, given in the same line of table 1, are pseudospin partners to each other. This is due to the fact that E_{nk} is affected by k only through $\gamma = k(k - 1)/r_0^2$. Because equation (34) is invariant under the mapping $k \rightarrow -k + 1$, any Dirac eigenstate with n and k will have a partner with n and $1 - k$, that is to say, the eigenstate with $E_{n,k}$ has a pseudospin partner with $E_{n,1-k}$.

Second, if we set $x = 0$ in equation (11), that is to say $r = r_0$, we get $D_0 + D_1 + D_2 = 1$. This relation can also be obtained from equation (12). Furthermore, from equation (12) we know that D_i ($i = 0, 1, 2$) is determined only by α , so, with fixed D, μ, C, n and k , we only need to specify any two of r_0, a and α to determine an eigenstate.

Third, perhaps because of a little mistake in the calculation from equation (36) to equation (37) in [11], the author concluded that there is only a negative bound state solution for $C \leq -10 \text{ fm}^{-1}$, and for $C > -10 \text{ fm}^{-1}$ there are no bound state solutions for the exact pseudospin symmetry Morse potential. Contrarily, our calculation shows that in the case of

Table 2. Relativistic energy eigenvalues in units of fm^{-1} of the pseudospin-symmetry Morse potential using the exact quantization rule method for various values of n and k . For a special case, $C = 0$.

\tilde{l}	$n, k < 0$	(l, j)	$E_{n,k < 0}$	$n - 1, k > 0$	$(l + 2, j + 1)$	$E_{n-1,k > 0}$
1	1, -1	(1s _{1/2})	-0.006 4899	0, 2	(0d _{3/2})	-0.006 4899
2	1, -2	(1p _{1/2})	-0.016 1835	0, 3	(0f _{5/2})	-0.016 1835
3	1, -3	(1d _{1/2})	-0.026 2288	0, 4	(0g _{7/2})	-0.026 2288
4	1, -4	(1f _{1/2})	-0.034 3246	0, 5	(0h _{9/2})	-0.034 3246
1	2, -1	(2s _{1/2})	-0.007 0456	1, 2	(1d _{3/2})	-0.007 0456
2	2, -2	(2p _{3/2})	-0.019 2957	1, 3	(1f _{5/2})	-0.019 2957
3	2, -3	(2d _{5/2})	-0.034 7132	1, 4	(1g _{7/2})	-0.034 7132
4	2, -4	(2f _{7/2})	-0.051 5127	1, 5	(1h _{9/2})	-0.051 5127

$C > -10 \text{ fm}^{-1}$, some negative bound state solutions for the exact pseudospin symmetry Morse potential can still exist. As an example, we tabulate the numerical solution of equations (32)–(34) in table 2, with the same parameters D, r_0, a, μ, D_0, D_1 and D_2 as in table 1, but for $C = 0$.

5. Conclusions

By using the exact quantization rule we have shown an alternative method to obtain the energy eigenvalues of the Dirac equation for the rotational Morse potential with the Pekeris approximation. The main results of this paper are the energy eigenvalues given by equations (32)–(34). Some details of the energy eigenvalues are discussed. We stress that some negative bound state solutions for the exact pseudospin symmetry Morse potential can still exist when $C > -10 \text{ fm}^{-1}$. The advantage of the exact quantization rule is that it gives the eigenvalues through the calculation of two integrations and solving of the resulting algebraic equation. The method presented here is a systematic one and is very efficient and practical. It is worth extending this method to the solutions of other non-relativistic or relativistic interaction problems.

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